Transformation Cookbook

Julian Straub
jstraub@csail.mit.edu

December 1, 2016

Contents

1 Introduction

This is a collection of facts and formulas for rigid body transformations. The aim is to establish a reference for practitioners, beginners, and professionals.

This is work in progress. Please bring any errors or suggestions to my attention via email at `jstraub@csail.mit.edu`.

2 Notation

$R \in \mathbb{R}^{3\times 3}$	rotation matrix
$q \in \mathbb{R}^4$	quaternion vector
$t \in \mathbb{R}^3$	translation vector
$T \in \mathbb{R}^{4\times 4}$	transformation matrix
$p \in \mathbb{R}^3$	point in 3D
$\theta \in [-\pi, \pi]$	angle of rotation
$\omega \in \mathbb{S}^2$	axis of rotation
$\alpha \in [0, \pi]$	angle between two rotations

- ${}^B\cdot_A$: transformation ${}^BT_A$, rotation ${}^BR_A$, translation ${}^Bt_A$ from reference frame A to reference frame B
- p_A: coordinates of point p in coordinate frame A
- skew operator $[\cdot]_\times$: construct skew symmetric matrix from vector

$$W = [w]_\times = \begin{pmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{pmatrix} = w_1 G_1 + w_2 G_2 + w_3 G_3 \tag{1}$$

$$= w_1 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} + w_2 \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} + w_3 \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} . \tag{2}$$

- vee operator $^\vee$: inverse of skew operator — extract vector from skew symmetric matrix

$$W^\vee = w = \begin{pmatrix} -W_{23} \\ W_{13} \\ -W_{12} \end{pmatrix} \in \mathbb{R}^3 . \tag{3}$$

- homogeneous coordinates: sometimes notation becomes easier by working in homogeneous coordinates. This involves increasing the dimension of a vector by 1

$$\hat{p} = \begin{pmatrix} p \\ 1 \end{pmatrix} \tag{4}$$

3 Rotation

Rotation is a fundamental part of a rigid body transformation. It can be described in different ways: rotation matrices, quaternions, and axis and angle.

In the following I will introduce these different representations and highlight their connections.

3.1 Rotation matrices SO(3)

$$R \in \mathbb{R}^{3\times 3} \quad (5)$$

$$\det(R) = 1 \quad (6)$$

$$R^T R = \mathrm{I} \quad (7)$$

$$R^{-1} = R^T \quad \text{inverse} \quad (8)$$

$${}^C R_A = {}^C R_B {}^B R_A \quad \text{composition} \quad (9)$$

$$R = U \begin{pmatrix} 1, 0 & 0 \\ 0 \ 1 & 0 \\ 0 \ 0 & det(UV^T) \end{pmatrix} V^T\,,\ \tilde{R} = USV^T \quad \text{rectification} \quad (10)$$

$$p_B = {}^B R_A p_A \quad \text{rotation} \quad (11)$$

$$\alpha = \arccos \tfrac{1}{2}(\mathrm{trace}(R^T R') - 1) \quad \text{distance} \quad (12)$$

3.1.1 Lie group and Lie algebra structure

- Exponential map $\mathrm{Exp} : \mathrm{so}(3) \to \mathrm{SO}(3)$:

$$\mathrm{Exp}(W) = \mathrm{I} + \tfrac{\sin(\theta)}{\theta} W + \tfrac{1-\cos(\theta)}{\theta^2} W^2\,,\ \theta = ||w||_2 \quad (13)$$

$$\mathrm{Exp}(\xi) = \mathrm{Exp}([\xi]_\times) \quad (14)$$

- Logarithm map $\mathrm{Log} : \mathrm{SO}(3) \to \mathrm{so}(3)$:

$$\mathrm{Log}(R) = \tfrac{\theta}{2\sin(\theta)} \left(R - R^T\right)\,,\ \theta = \arccos \tfrac{1}{2}(\mathrm{trace}(R) - 1) \quad (15)$$

$$\xi = \mathrm{Log}(R)^\vee \quad (16)$$

- Generators of so(3):

$$G_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}\,,\ G_y = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}\,,\ G_z = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (17)$$

$$[\xi]_\times = G_x \xi_x + G_y \xi_y + G_z \xi_z \quad (18)$$

The exp and log map can be understood as mapping between the rotation manifold and its tangent space around the identity rotation. If we want to map a rotation into a different tangent space around R_A we simply compute:

$$\mathrm{Exp}_{R_A}(W) = R_A \mathrm{Exp}(W) \quad (19)$$

$$\mathrm{Log}_{R_A}(R) = R_A \mathrm{Log}(R_A^T R) \quad (20)$$

$$\theta = \arccos \tfrac{1}{2}(\mathrm{trace}(R_A^T R) - 1) \quad (21)$$

3.1.2 Conversions from other representations

$$R = \begin{pmatrix} q_w^2 + q_x^2 - q_y^2 - q_z^2 & 2(q_x q_y - q_w q_z) & 2(q_x q_z + q_w q_y) \\ 2(q_x q_y + q_w q_z) & q_w^2 - q_x^2 + q_y^2 - q_z^2 & 2(q_y q_z - q_w q_x) \\ 2(q_x q_z - q_w q_y) & 2(q_y q_z + q_w q_x) & q_w^2 - q_x^2 - q_y^2 + q_z^2 \end{pmatrix} \text{[Hor87]} \quad (22)$$

$$R = \mathrm{Exp}(\theta[\omega]_\times) \quad (23)$$

(24)

3.1.3 Derivatives

$$\frac{\partial}{\partial \xi} \mathrm{Exp}(\xi) R p \Big|_{\xi=0} = -[Rp]_\times \tag{25}$$

$$\frac{\partial}{\partial \xi} R \mathrm{Exp}(\xi) p \Big|_{\xi=0} = -R[p]_\times \tag{26}$$

$$\frac{\partial}{\partial \xi_x} \frac{\partial}{\partial \xi_y} \mathrm{Exp}(\xi) R p \Big|_{\xi=0} = \tfrac{1}{2}(G_x G_y + G_y G_x) R p \tag{27}$$

$$\frac{\partial}{\partial \xi_x} \frac{\partial}{\partial \xi_y} R \mathrm{Exp}(\xi) p \Big|_{\xi=0} = \tfrac{1}{2} R (G_x G_y + G_y G_x) p \tag{28}$$

where $\xi = \theta\omega$.

3.2 Quaternions

Quaternions are 4D extensions of the imaginary numbers $q = q_w + iq_x + jq_y + kq_z$. The unit length quaternions can be used to describe 3D rotations. For our purposes we think of unit quaternions $q = (q_w, q_{xyz})$ as points lying on the sphere in 4D, $\mathbb{S}^3$. $\mathbb{S}^3$ is a double cover of the rotation space. Hence it is sufficient to only consider the upper half sphere in 4D to cover the space of rotations completely.

$$q \in \mathbb{R}^4 \tag{29}$$

$$\|q\|_2 = 1 \tag{30}$$

$$q^{-1} = (q_w, -q_{xyz}) \qquad \text{inverse} \tag{31}$$

$$^{C}q_A = {^{C}q_B} \circ {^{B}q_A} = r \circ q = \begin{pmatrix} r_w q_w - r_x q_x - r_y q_y - r_z q_z \\ r_w q_x + r_x q_w + r_y q_z - r_z q_y \\ r_w q_y - r_x q_z + r_y q_w + r_z q_x \\ r_w q_z + r_x q_y - r_y q_x + r_z q_w \end{pmatrix} \qquad \text{composition [Hor87]} \tag{32}$$

$$p_B = {^{B}q_A} \circ p_A \qquad \text{rotation} \tag{33}$$

$$q \circ p = (-q) \circ p \tag{34}$$

$$q = \tfrac{q'}{\|q'\|_2} \qquad \text{rectification} \tag{35}$$

$$\alpha = 2 \arctan \tfrac{\|\Delta q_{xyz}\|_2}{\Delta q_w}, \; \Delta q = q^{-1} \circ q' \qquad \text{distance} \tag{36}$$

3.2.1 Conversions from other representations

$$q = (\cos \tfrac{\theta}{2}, \sin \tfrac{\theta}{2} \omega) \tag{37}$$

3.3 Axis Angle (AA)

Axis ω and angle θ

$$\theta \in [-\pi, \pi] \tag{38}$$
$$\|\omega\|_2 = 1 \tag{39}$$
$$[\theta\omega]_\times \in \mathrm{so}(3) \tag{40}$$

Axis angle rotations can neither be composed directly nor can they directly transform 3D points.

3.3.1 Conversions from other representations

$$\theta = 2 \arccos q_w \tag{41}$$
$$\theta = 2 \arctan \tfrac{\|q_{xyz}\|_2}{q_w} \text{ (numerically more stable)} \tag{42}$$
$$\omega = \tfrac{q_{xyz}}{\|q_{xyz}\|_2} \tag{43}$$
$$\theta\omega = \mathrm{Log}(R)^\vee \tag{44}$$
$$\theta = \arccos \tfrac{1}{2}(\mathrm{trace}(R) - 1) \tag{45}$$

4 Rigid body transformations SE(3)

Rigid body transformation is a composition if a rotation and a translation.

$$T = \begin{pmatrix} R & t \\ 0 & 1 \end{pmatrix} \tag{46}$$
$$T^{-1} = \begin{pmatrix} R^T & -R^T t \\ 0 & 1 \end{pmatrix} \quad \text{inverse} \tag{47}$$
$${}^C T_A = {}^C T_B {}^B T_A = \begin{pmatrix} {}^C R_B {}^B R_A & {}^C R_B {}^B t_A + {}^C t_B \\ 0 & 1 \end{pmatrix} \quad \text{composition} \tag{48}$$
$$\hat{p}_A = {}^A T_B \hat{p}_B \quad \text{transformation} \tag{49}$$

The homogeneous coordinate representation above has a nice clean notation but in practical implementations one usually does not want to spend memory on storing the constant 0s and the 1 in the fourth row of T.

$$T = \{R, t\} \tag{50}$$
$$T^{-1} = \{R^T, -R^T t\} \quad \text{inverse} \tag{51}$$
$${}^C T_A = {}^C T_B {}^B T_A = \{{}^C R_B {}^B R_A, {}^C R_B {}^B t_A + {}^C t_B\} \quad \text{composition} \tag{52}$$
$$p_A = {}^A R_B p_B + {}^A t_B \quad \text{transformation} \tag{53}$$

Alternatively we can use a unit Quaternion to represent the rotation.

$$T = \{q, t\} \tag{54}$$
$$T^{-1} = \{q^{-1}, -(q^{-1} \circ t)\} \quad \text{inverse} \tag{55}$$
$${}^C T_A = {}^C T_B {}^B T_A = \{{}^C q_B \circ {}^B q_A, {}^C q_B \circ {}^B t_A + {}^C t_B\} \quad \text{composition} \tag{56}$$
$$p_A = {}^A q_B \circ p_B + {}^A t_B \quad \text{transformation} \tag{57}$$

5 Bibliography

References

[Hor87] Berthold KP Horn. Closed-form solution of absolute orientation using unit quaternions. *JOSA A*, 4(4):629–642, 1987.